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A direct construction of the Wiener measure on $\mathbf{C}[0, \infty)$

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ABSTRACT
Our construction of the Wiener measure on $\mathbf{C} = \mathbf{C}[0, \infty)$ consists in first defining a set function φ on the class of all compact sets based on certain n-dimensional normal distributions, $n = 1, 2, \ldots$ using the structural relation at (2) below. This structural relation, discovered by the first author, is recorded in his book (2013) on page 130. We then define a measure μ on the Borel σ-field of subsets of $\mathbf{C}$ which is the Wiener measure. This is done via a similar construction of the Wiener measure on $\mathbf{C}_a = \mathbf{C}[0, a)$ where $a > 0$ is an arbitrary real number.
The traditional way is to first construct the Brownian Motion process (BMP) and then, by proving it is a measurable mapping into $(\mathbf{C}, \mathfrak{C}_\infty)$, call the measure induced by the BMP on $\mathbf{C}$ the Wiener measure. In the present paper, we define the Wiener measure directly.

KEYWORDS
Construction of Wiener measure, Brownian Motion Process, A structural relation.

AMS Subject Classification 60J65; 60G15.

1. Introduction

Construction of the Wiener process is discussed by many authors and the discussion invariably starts with the construction first of the Brownian Motion Process (BMP) on a probability space. The BMP is studied for its properties and then is proved to be a measurable mapping into $\mathbf{C}[0, 1]$ space endowed with the uniform metric and the resulting Borel σ-field $\mathfrak{C}_1$. Call the measure induced by the BMP on $\mathbf{C}[0, 1]$ the Wiener measure. We refer to chapter 2 in [1].
The aim of the present paper is to reverse the procedure and construct the Wiener measure directly using elementary measure theory and the structural relation given at (2) below. In Pakshirajan, R. P. and Sreehari, M. An elementary construction of the Wiener measure, arXiv:2011.05584v1 [math.PR] 11 November 2020, the authors presented the construction of the Wiener measure on $\mathbf{C}[0, 1]$.
Let $a > 0$ be arbitrary but fixed. Let $\mathbf{C}_a = \mathbf{C}[0, a]$ denote the space of real valued continuous functions defined on $[0, a]$, all vanishing at 0 and endowed with the norm

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$\|x\|_a = \sup\limits_{0\le t\le a} |x(t)|,\ x \in \mathbf{C}_a$. Let $\rho_a(x,\ y)$ denote the associated metric. Define another norm : $\|x\|_a^* = \sup\limits_{0\le s,\ t\le a} |x(t) - x(s)|$. The associated metric will be denoted by ρ_a^*. Since $\|x\|_a \le \|x\|_a^* \le 2\|x\|_a$, the two norms induce the same topology in $\mathbf{C}_a$ and determine the same Borel σ-field.
Let $T_n = \{\frac{ak}{2^n},\ k = 0,\ 1,\ 2, \dots,\ 2^n\}; T = \bigcup\limits_{n=1}^{\infty} T_n$ and note that T is a countable dense subset of the interval $[0,\ a]$. For $x \in \mathbf{C}_a$, define

$$\wp_n x = \Big(x(\frac{a}{2^n}),\ x(\frac{2a}{2^n}) - x(\frac{a}{2^n}), \dots, x(\frac{a2^n}{2^n}) - x(\frac{a(2^n-1)}{2^n})\Big). \tag{1}$$

This maps $\mathbf{C}_a$ into R^{2^n}. Assume R^{2^n} is endowed with the usual metric and denote the resulting Borel σ-field by $\mathfrak{R}^{2^n}$. We note $\wp_n$ is a continuous map and hence is $\mathfrak{C}_a$ measurable. We prescribe the distribution of the vector variable $\wp_n$ to be the multivariate normal distribution with independent components, each component with zero mean and variance $\frac{a}{2^n}$. i.e., it is the joint distribution of $(\sqrt{\frac{a}{2^n}}\xi_k,\ 1 \le k \le 2^n)$ where the ξ_ks are independent standard normal variables.
Denote by ν_n the measure on $\mathfrak{R}^{2^n}$ by this distribution.
Let α_n denote the measure generated on the sub σ-field $\wp_n^{-1}(\mathfrak{R}^{2^n})$ by the mapping $\wp_n$.
All sets considered below are members of $\mathfrak{C}_a$.
Let $K \subset \mathbf{C}_a$ be compact. Then the following structural relation holds: (ref. pp 130-131 in [2].)

$$K = \cap_{n=1}^{\infty} \wp_n^{-1} \wp_n K. \tag{2}$$

To make for seamless reading we present here a proof of (2).
That $K \subset \cap_{n=1}^{\infty} \wp_n^{-1}\wp_n K$ is obvious. Now to establish the reverse inclusion, let x be an arbitrary member of the right side. Hence for every n, $x \in \wp_n^{-1}\wp_n K$. There exists therefore $y_n \in K$ such that $\wp_n x = \wp_n y_n$. Since K is compact, sequence (y_n) contains a convergent subsequence, say, (y_m) converging to, say, $y_0 \in K$ in the metric $\rho_{\mathfrak{a}}$. This implies $y_m(t) \to y_0(t)$ for all $t \in [0, a]$. Fix r and let $1 \le j \le 2^r$. Let $m > r$. The relation $\wp_m x = \wp_m y_m$, is equivalent to the relation $\wp_m x = \wp_m y_m$ in the sense that given $\wp_m x \in R^{2^m}$ the point $\wp_m x$ is uniquely determined and conversely through a linear transformation. Here $\wp_m x = \Big(x(\frac{a}{2^m}),\ x(\frac{2a}{2^m}),\ ...,\ x(\frac{a2^m}{2^m})\Big)$. We get $x(\frac{aj}{2^r}) = y_m(\frac{aj}{2^r})$. Take limit as $m \to \infty$, and get $x(\frac{aj}{2^r}) = y_0(\frac{aj}{2^r})$. Thus for every $u \in T,\ x(u) = y_0(u)$. Since T is dense in $[0,\ a]$ and since $x,\ y_0$ are continuous functions, it follows that $x(t) = y(t)$ for all $t \in [0,\ a]$. Thus $x \in K$ and the proof is complete.
Note that this inclusion is true for any set K and not only for compact sets.

Theorem 1.1. *For any $A \in \mathfrak{C}_a,\ \alpha_n(\wp_n^{-1}\wp_n(A)),\ n = 1,\ 2, \dots$ is a monotonic decreasing sequence of numbers.*

Proof.

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$$\alpha_{n+1}\big(\wp_{n+1}^{-1}\wp_{n+1}(A)\big) = \int_{\wp_{n+1}(A)} d\nu_{n+1} \leq \int\limits_{\wp_n(A)\times R} d\nu_n \, d\beta_{n+1}$$
$$\leq \int\limits_{\wp_n(A)} d\nu_n \leq \alpha_n(\wp_n^{-1}\wp_n(A))$$

where β_{n+1} is the distribution function of a normal variable and $\nu_n = \alpha_n\wp_n^{-1}$. □

Define set function φ on the compact sets K of $\mathbf{C}_a$:

$$\varphi(K) = \lim_{n\to\infty} \alpha_n\big(\wp_n^{-1}\wp_n(K)\big). \tag{3}$$

Note

$$\varphi(K) \leq 1; \quad \varphi(\emptyset) = 0. \tag{4}$$

□

Theorem 1.2. *Let $K_1,\ K_2$ be compact sets such that $\varphi(K_1 \cap K_2) = 0$. Then $\varphi(K_1 \cup K_2) = \varphi(K_1) + \varphi(K_2)$.*

Proof.
$\wp_n(K_1 \cup K_2) = \wp_n(K_1) \cup \wp_n(K_2)$
$\wp_n^{-1}\wp_n(K_1 \cup K_2) = \wp_n^{-1}\wp_n(K_1) \cup \wp_n^{-1}\wp_n(K_2)$.
Hence

$$\alpha_n\big(\wp_n^{-1}\wp_n(K_1 \cup K_2)\big) = \alpha_n\big(\wp_n^{-1}\wp_n(K_1)\big) + \alpha_n\big(\wp_n^{-1}\wp_n(K_2)\big)$$
$$-\alpha_n\big(\wp_n^{-1}\wp_n(K_1) \cap \wp_n^{-1}\wp_n(K_2)\big)$$

since α_n is a measure. Now, $\wp_n^{-1}\wp_n(K_1)\cap\wp_n^{-1}\wp_n(K_2) = \wp_n^{-1}\wp_n(K_1\cap K_2)$. Since $K_1\cap K_2$ is a compact set and since $\varphi(K_1 \cap K_2) = 0$, $\alpha_n\big(\wp_n^{-1}\wp_n(K_1 \cap K_2)\big) < \varepsilon$ for all n large. Taking limits as $n \to \infty$ and then as $\varepsilon \to 0$ in the inequalites

$$\alpha_n\big(\wp_n^{-1}\wp_n(K_1)\big) + \alpha_n\big(\wp_n^{-1}\wp_n(K_2)\big) - \varepsilon \leq \alpha_n\big(\wp_n^{-1}\wp_n(K_1 \cup K_2)\big)$$
$$\leq \alpha_n\big(\wp_n^{-1}\wp_n(K_1)\big) + \alpha_n\big(\wp_n^{-1}\wp_n(K_2)\big)$$

we complete the proof of the claim. □

Remark 1. We have the following observations from the earlier discussion:
a) φ is finitely additive on the collection of compact sets.
b) $0 \leq \varphi(K) \leq 1$ for all compact sets K.
c) If K_1, K_2 are compact sets and $K_1 \subset K_2$ then from (3), $\varphi(K_1) \leq \varphi(K_2)$.

Definition 1.3. We call a set, in a topological space, a boundary set if it is a closed set with a null interior. The boundary of a set A (i.e., $\overline{A} \sim Int\ A$) will be denoted by ∂A.

Here we denote the closure of a set A by $\overline{A}$ and interior of a set A by $IntA$. Further complement of a set A is denoted by A'.
We note that the boundary of a set is a boundary set.

Theorem 1.4. *If $K_1,\ K_2$ are compact subsets with $K_1 \subset K_2,$ and $\varphi(\partial K_1) = 0$, then $\varphi(K_2 \cap \bar{K}_1') = \varphi(K_2) - \varphi(K_1)$.*

Proof. $\bar{K}_1' \cap K_2$ is a compact set. $K_1 \cap \{\bar{K}_1' \cap K_2\} = \partial K_1$. Since $\varphi(\partial K_1) = 0$, Theorem 1.2 applies and we get $\varphi(K_2) = \varphi\big(K_1 \cup \{\bar{K}_1' \cap K_2\}\big) = \varphi(K_1) + \varphi\big(\bar{K}_1' \cap K_2\big)$, as was to be proved. □

We now discuss some limiting properties of $\varphi(K_n)$.

Theorem 1.5. *(i) Let $K_n, n \geq 1$, be compact sets such that K_n decreases to K. Then $\varphi(K_n)$ decreases to $\varphi(K)$.*
(ii) Suppose $K, K_n, n \geq 1$ are compact subsets, K_n increases to K and $\varphi(\partial K_n) = 0$. Then $\varphi(K_n)$ increases to $\varphi(K)$.
(iii) Let $K,\ K_m,\ m \geq 1$ be compact sets, K_m increases to K and $\varphi(K_m) = 0$. Then $\varphi(K) = 0$.

Proof. (i) Since the sequence $(\varphi(K_n))$ is monotonic decreasing, it is enough to show that, given $\varepsilon > 0$, there exists K_N such that $\varphi(K_N) < \varphi(K) + \varepsilon$.
We note that K is compact. Hence given $\varepsilon > 0$, we can find $r \geq 1$ such that

$$\varphi(K) > \alpha_\ell\big(\wp_\ell^{-1}\wp_\ell(K)\big) - \varepsilon \tag{5}$$

for all $\ell \geq r$. Since K_n decreases to K, for all $\ell \geq 1$ we have that $\wp_\ell^{-1}\wp_\ell(K_n)$ decreases to $\wp_\ell^{-1}\wp_\ell(K)$.
For fixed ℓ we then have that, as $n \to \infty$, $\alpha_\ell\big(\wp_\ell^{-1}\wp_\ell(K_n)\big)$ decreases to $\alpha_\ell\big(\wp_\ell^{-1}\wp_\ell(K)\big)$. Take $\ell = r$. We can find $N = N(r)$ large such that $\alpha_r\left(\wp_r^{-1}\wp_r(K_N)\right) < \alpha_r\left(\wp_r^{-1}\wp_r(K)\right) + \varepsilon$. This, together with (5), yields

$$\varphi(K) + \varepsilon > \alpha_r\left(\wp_r^{-1}\wp_r(K_N)\right) - \varepsilon > \varphi(K_N) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the claim follows.
(ii) Define $E_n = K \cap \bar{K}_n'$ and note by Theorem 1.4 that $\varphi(E_n) = \varphi(K) - \varphi(K_n)$. Now, the E_ns are compact sets and E_n decreases to $\emptyset$. Hence by part (i) above, $\varphi(E_n)$ decreases to 0. i.e., $\varphi(K_n) \to \varphi(K)$, as was to be proved.
(iii) Claim immediate from part(ii) above. □

2. THE WIENER MEASURE on $\mathbf{C}_a$.

In this Section we introduce a new set function in terms of φ on the Borel σ-field $\mathfrak{C}_a$ of subsets of $\mathbf{C}_a$ and study its properties to show that it is indeed the Wiener measure. For arbitrary measurable sets $A \in \mathfrak{C}_a$ define

$$\mu(A) = \sup_{K \subset A,\ K \text{ compact}} \varphi(K). \tag{6}$$

At the outset we observe that for compact sets K, $\mu(K) = \varphi(K)$ and hence all the properties noted in the previous Section for φ also hold for μ. Further the definition implies (i) that if $A \subset B, A, B \in \mathfrak{C}_a$ then $\mu(A) \leq \mu(B)$ and (ii) that there exists an increasing sequence (K_n) of compact sets, $K_n \subset A$ such that $\mu(A) = \lim_{n\to\infty} \mu(K_n)$. The sets K_n can be chosen to be monotonic increasing.

Remark 2. This does not mean that K_n increases to A. i.e., $\bigcup_{n=1}^{\infty} K_n$ can be a proper subset of A. To see this, take $v \in \mathbf{C}, \|v\| = 1$. Let $K_n = \{\lambda v,\ 0 \leq \lambda \leq 1 - \frac{1}{n}\}$ and $A = \{\lambda v,\ 0 \leq \lambda \leq 1\}$. However, if $K_n = \{\lambda v,\ 0 \leq \lambda \leq 1 - \frac{1}{n}\} \cup \{v\}$ then both K_n and A are compact and K_n increases to A.

We next discuss further properties of μ that enable us to claim that μ is indeed a probability measure.

Theorem 2.1. *(i) If $A \subset B$, then $\mu(A) \leq \mu(B)$.*
(ii)If $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Proof. (i) Immediate from the definition of μ at (6)
(ii) Let $E \subset A,\ F \subset B$ be compact sets such that for a given $\varepsilon > 0 \quad \mu(E) > \mu(A) - \varepsilon$ and $\mu(F) > \mu(B) - \varepsilon$. We have, from Theorem 1.2
$\mu(E) + \mu(F) = \mu(E \cup F) \leq \mu(A \cup B)$ since ε is arbitrary.
Thus $\mu(A \cup B) \geq \mu(A) + \mu(B)$. It remains to be shown that $\mu(A \cup B) \leq \mu(A) + \mu(B)$.
Given $\varepsilon > 0$, we can find a compact set $K,\ \ K \subset A \cup B$ such that
$\mu(A \cup\ B) - \varepsilon < \mu(K)$.
Case 1. The distance $d(A,\ B) = q > 0$.
Consider an arbitrary sequence (x_n) in $K \cap A$. Since it is a sequence in K, it contains a convergent subsequence, converging to, say, x_0. This x_0 has to be in $K \cap A$ or in $K \cap B$. Since the sequence lies in $K \cap A$ and since $d(K \cap A,\ K \cap B) \geq q > 0$, we conclude $x_0 \in K \cap A$. Thus we see that every sequence in $K \cap A$ contains a convergent subsequence converging to a point in $K \cap A$. This means $K \cap A$ is a compact set. Similarly, $K \cap B$ is a compact set. Summarising, we conclude that every compact subset of $A \cup B$ is the union of a compact subset E of A and a compact subset F of B. We get $\mu(A \cup\ B) - \varepsilon < \mu(K) = \mu(E \cup F) = \mu(E) + \mu(F) \leq \mu(A) + \mu(B)$. That $\mu(A \cup B) \leq \mu(A) + \mu(B)$ is now immediate.
Case 2. $d(A,\ B) = 0$.
This case assumption implies that $Q = \bar{A} \cap \bar{B} \neq \emptyset$. Again in this case one or both the sets $K \cap A,\ K \cap B$ can fail to be compact. Since the other case admits to being similarly argued, let us assume that neither of the two sets is compact. $K \subset A \cup B$ can not be compact if any convergent sequence in it converges to a point outside K. i.e., if convergent sequences in $K \cap A$ or in $K \cap B$ converge to points outside these sets. Thus K can be a compact subset only if $E = K \cap A$ and $F = K \cap B$ are compact. And the arguments and the conclusion in case 1 hold.
With this the proof is complete. □

Remark 3. Immediate consequences of Theorem 2.1 are :
a) If $A_k,\ 1 \leq k \leq n$ is any collection of n events, then $\mu(\bigcup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} \mu(A_k)$ and if

the events A_n are mutually exclusive equality holds.
(b) If $A \subset B$, then $\mu(B \sim A) = \mu(B) - \mu(A)$.

Our next result shows that μ is monotone.

Theorem 2.2. *(i) If A_n decreases to A, then $\mu(A_n)$ decreases to $\mu(A)$.*
(ii)If A_n increases to A, then $\mu(A_n)$ increases to $\mu(A)$.

Proof.
(i) By Theorem 2.1 we note that the hypothesis implies B_n decreases to $\emptyset$ where $B_n = A_n \cap A'$. We refer to Remark 3(a) and claim that it is enough to show that $\mu(B_n) \to 0$.
Find compact sets $K_n \subset B_n$ such that $\mu(B_n) - \mu(K_n) < \frac{\varepsilon}{2^n}$. Define $Q_n = \cap_{j=1}^n K_j$, Note that $Q_n \subset B_n$, that Q_n is a compact set and that Q_n decreases to $\emptyset$, By Theorem 1.5 and the fact that $\mu(\phi) = 0$ by (4), it then follows that $\mu(Q_n) \to 0$. Further by Remark 3(a)

$$\mu(B_n) - \mu(Q_n) = \mu(B_n \cap Q_n') \ = \mu\big(B_n \cap \{\bigcup_{j=1}^n K_j'\}\big) = \mu\big(\bigcup_{j=1}^n (B_n \cap K_j')\big)$$
$$\leq \sum_{j=1}^n \mu(B_n \cap K_j') \leq \sum_{j=1}^n \mu(B_j \cap K_j') \leq \sum_{j=1}^n \{\mu(B_j) - \mu(K_j)\} \leq \sum_{j=1}^n \frac{\varepsilon}{2^j} < \varepsilon$$

for all n. Here we used the Remark 3 and the fact that B_n is decreasing. Collecting the results, we conclude $\mu(B_n) \to 0$, thus completing the proof of this part.
(ii) That $\mu(A_n)$ is increasing is true follows from Theorem 2.1(i). Since $A \cap A_n'$ decreases to $\emptyset$, part (i) applies and we have $\mu(A_n \cap A') \to 0$. Now by Remark 3(b), this gives $\mu(A_n) \to \mu(A)$. □

Theorem 2.3. *μ defined at (6) is a probability measure.*

Proof.
Let $A_n \in \mathfrak{C}_a,\ n \geq 1$ be a sequence of mutually exclusive events. Let $A = \bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty B_n$ where $B_n = \bigcup_{k=1}^n A_k$. Since B_n increases to A, Remark 3(a) applies and then we have

$$\mu(A) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{k=1}^n \mu(A_k) = \sum_{k=1}^\infty \mu(A_k).$$

i.e., μ on $\mathfrak{C}_a$ is countably additive. Since $\mu(A) \geq 0$ for $A \in \mathfrak{C}$ it follows that μ is a probability measure if we show that $\mu(\mathbf{C}_a) = 1$.
Let $T = \cup_{n=1}^\infty T_n,\ T_n = \{t_k,\ 1 \leq k \leq 2^n\}$ where $t_k = t_{k,n} = \frac{ak}{2^n}$ and note that T is a countable dense subset of the interval $[0,\ a]$. Let $S_m = \{x:\ x \in \mathbf{C}_a,\ \|x\| \leq m\}$. We note $S_m = \{x:\ x \in \mathbf{C}_a,\ \sup_{t\in T} |x(t)| \leq m\} = \cap_{n=1}^\infty B_{n,m} = \lim_{n\to\infty} B_{n,m}$ where $B_{n,m} = \{x:\ x \in \mathbf{C}_a,\ \sup_{t\in T_n} |x(t)| \leq m\} = \cap_{t\in T_n}\{x:\ x \in \mathbf{C}_a,\ |x(t)| \leq m\}$.
We note that S_m increases to $\mathbf{C}_a$. Recall that, given $\varepsilon > 0$, we can find $A_m \subset S_m,\ A_m$ compact such that $\mu(S_m) - \mu(A_m) < \varepsilon$. Write
$\wp_{T_n} x = \Big(x((\frac{a}{2^n}),\ x(\frac{2a}{2^n}) - x(\frac{a}{2^n}),\ x(\frac{3a}{2^n}) - x(\frac{2a}{2^n}),\ ...,\ x(\frac{a2^n}{2^n}) - x(\frac{a(2^n-1)}{2^n})\Big)$.

If $K \subset \mathbf{C}_a$ is a compact set, then arguing as in the proof of Theorem 1.1 we get that $\wp_{T_n}^{-1}\wp_{T_n}K$ decreases to K. Hence given $\varepsilon > 0$, we can find N such that for all $n \geq N$,
$\mu(A_m) + \varepsilon > \alpha_{T_n}\wp_{T_n}^{-1}\big(\wp_{T_n}S_m\big) = P\big(\frac{a^{1/2}}{2^{n/2}} \max_{1\leq j\leq 2^n} |\xi_j| \leq m\big)$
where the ξs are independent standard normal variables. Hence
$\mu(A_m) + \varepsilon > \big(P(|\xi| \leq \frac{2^{n/2}\, m}{a^{1/2}})\big)^{2^n} = \big(1 - P(|\xi| > \frac{2^{n/2}\, m}{a^{1/2}})\big)^{2^n} \geq \big(1 - \frac{a\mathbb{E}|\xi|^2}{m^2\, 2^n}\big)^{2^n}$
leading to $\mu(A_m) + \varepsilon \geq e^{-(a/m^2)}$.
This implies by Theorem 2.2(ii),
$\mu(\mathbf{C}_a) = \lim_{m\to\infty} \mu(S_m) \geq \lim_{m\to\infty} \mu(A_m) \geq \lim_{m\to\infty} e^{-a/m^2} - \varepsilon \geq 1 - \varepsilon.$
Since $\varepsilon > 0$ is arbitrary we get $\mu(\mathbf{C}_a) = 1$. □

2.1. Alternate proof for $\mu(C_a) = 1$ which will be useful in Section 3.

Since $\mu(K) \leq 1$ for all compact sets, as noted in (4) and since
$\mu(\mathbf{C}_a) = \sup_{K\subset\mathbf{C}_a,\ K \text{ compact}} \mu(K)$, it follows that $\mu(\mathbf{C}_a) \leq 1$. So the proof will be complete if we show that $\mu(\mathbf{C}_a) \geq 1$. This we proceed to show.
Let

$$H_\alpha^0 = \{x : \sup_{0\leq s,t\leq 1;\ s\neq t} \frac{|x(t) - x(s)|}{|t - s|^\alpha} < \infty\}$$

and

$$H_{\alpha,a}^0 = \{x : \sup_{0\leq s,t\leq a;\ s\neq t} \frac{|x(t) - x(s)|}{|t - s|^\alpha} < \infty\}.$$

Let $\delta > 0$ be arbitrary. Then for $|s - t| < \delta$ and $x \in H_\alpha^0$ without loss of generality we have

$$|x(s) - x(t)| < |t - s|^\alpha < \delta^\alpha.$$

Then we have the following Theorem which in turn implies $\mu(\mathbf{C_a}) = \mathbf{1}$.

Theorem 2.4. *For* $0 < \alpha < 1,\ \ \mu(H_\alpha^0) = 1.$

Proof.
Take n large so that $\frac{a}{2^n} < \delta$. Then $|x(\dfrac{a(r+1)}{2^n}) - x(\dfrac{ar}{2^n})| \leq (\frac{a}{2^n})^\alpha$ for $r = 0, 1, \ldots, n-1$.
Since the μ measure of every compact subset of $\mathbf{C_a} \leq \mathbf{1}$ it follows that the μ measure any measurable subset of $\mathbf{C_a} \leq \mathbf{1}$ as well. Now since

$$H_{\alpha,a}^0 = \{x : \sup_{0\leq s,t\leq a;\ s\neq t} \frac{|x(t) - x(s)|}{|t - s|^\alpha} < \infty\}$$

is a measurable subset of $\mathbf{C_a}$ it follows that $\mu(H_{\alpha,a}^0) \leq 1$.
Set

$$\mathbb{S}_{\alpha,a}(\lambda) = \{x : \sup_{0<s,t\leq a;\ 0<|s-t|<1} \frac{|x(t) - x(s)|}{|s - t|^\alpha} \leq \lambda\}.$$

Then $\mathbb{S}_{\alpha,a}(\lambda)$ is a compact subset of $\mathbf{C_a}$ (see Appendix) and $\mu((\mathbb{S}_{\alpha,a}(\lambda))') < \varepsilon$ for all large λ depending on ε. i.e., $\lim_{n\to\infty} \nu_n(\wp_n(\mathbb{S}_{\alpha,a}(\lambda))')) < \varepsilon$.
Let

$$A_n = \{x : x \in H^0_{\alpha,a}; \max_{0\le r\le 2^n-1} |x(a(r+1)) - x(ar)| \le \frac{\lambda a^\alpha}{2^{n\alpha}}\}.$$

Note that for $x \in A_n$

$$\wp_n(x) = \left(x(\frac{a}{2^n}), x(\frac{2a}{2^n}) - x(\frac{a}{2^n}), \dots, x(\frac{a2^n}{2^n}) - x(\frac{a(2^n-1)}{2^n})\right).$$

We then have

$$\begin{aligned}
\mu(\mathbb{S}_{\alpha,a}(\lambda)) = \lim_{n\to\infty} \nu_n(\wp_n(\mathbb{S}_{\alpha,a}(\lambda))) &\ge \lim_{n\to\infty} \nu_n(\wp_n(\mathbb{S}_{\alpha,a}(\lambda) \cap A_n)) \\
&\ge \lim_{n\to\infty} [\nu_n(\wp_n A_n) - \nu_n(\wp_n(A_n \cap (\mathbb{S}_{\alpha,a}(\lambda))')] \\
&\ge \lim_{n\to\infty} [\nu_n(\wp_n A_n) - \nu_n(\wp_n((\mathbb{S}_{\alpha,a}(\lambda))')] \\
&\ge \lim_{n\to\infty} [\nu_n(\wp_n A_n) - \varepsilon] \\
\ge \lim_{n\to\infty} P(\max_{0\le r\le 2^n-1} |x(a(r+1)) - x(ar)| &\le \frac{\lambda a^\alpha}{2^{n\alpha}}) - \varepsilon \\
&\ge \lim_{n\to\infty} P(\max_{0\le r\le 2^n-1} |\xi_r| \le \frac{\lambda 2^{n(1-2\alpha)/2}}{a^{(1-2\alpha)}/2}) - \varepsilon \\
&= \lim_{n\to\infty} [P(|\xi| \le \frac{\lambda 2^{n(1-2\alpha)/2}}{a^{(1-2\alpha)}/2})]^{2^n} - \varepsilon \\
&= \lim_{n\to\infty} [1 - P(|\xi| > \frac{\lambda 2^{n(1-2\alpha)/2}}{a^{(1-2\alpha)/2}})]^{2^n} - \varepsilon \\
&\ge \lim_{n\to\infty} [1 - \frac{a}{2^n} \frac{E|\xi|^{2/(1-2\alpha)}}{\lambda^{2/(1-2\alpha)}}]^{2^n} - \varepsilon \\
&\ge e^{-\psi(\lambda)} - \varepsilon
\end{aligned}$$

by Chebyshev's inequality where ξ, ξ_k are independent standard normal rvs and $\psi(\lambda) = \frac{aE|\xi|^{2/(1-2\alpha)}}{\lambda^{2/(1-2\alpha)}} \to 0$ as $\lambda \to \infty$. Since ε is arbitrary from the above result we get $\mu(\mathbb{S}_{\alpha,a}(\lambda)) \ge 1$ and hence $\mu(H^0_\alpha) = 1$. This completes the proof of Theorem 2.4. □

Remark 4. (i) From the construction of μ_a, it is clear that if ν is a probability measure on $\mathfrak{C}_a$ and if its finite dimensional distributions (i.e., the distributions of the vector variables $(\pi_{t_1}, \pi_{t_2}, \dots, \pi_{t_k})$, for every choice of k and every choice of $(t_1, t_2, \dots, t_k)$ are the same as the corresponding ones of μ_a then $\nu \equiv \mu_a$. It follows now that μ is the Wiener probability measure.
(ii) The co-ordinate process $\{\pi_t, t \ge 0\}$ is known as the Brownian motion process.

3. Constructing the Wiener measure on $\mathbf{C}_\infty$.

Let $\mathbf{C}_\infty$ be the space of all the real continuous functions defined on $[0, \infty)$, all vanishing at 0 endowed with the metric

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0\le t\le n} |f(t) - g(t)|}{1 + \sup_{0\le t\le n} |f(t) - g(t)|}$$

Further on $\mathbf{C}_r$ define the metric

$$d_r(f, g) = \sum_{n=1}^{r} \frac{1}{2^n} \frac{\sup_{0\le t\le n} |f(t) - g(t)|}{1 + \sup_{0\le t\le n} |f(t) - g(t)|}.$$

Let $\mathfrak{C}_\infty$ denote the Borel σ-field of $(\mathbf{C}_\infty,\ d)$ for $f, g \in \mathbf{C}_\infty$. Note that $(\mathbf{C}_\infty,\ d)$ is a complete and separable space and the metrics $d(f, g)$ and $d_r(f, g)$ are bounded by 1. Define mapping Q_r as $Q_r f(t) = f(t), 0 \le t \le r$ for $r \ge 1$ and $f \in \mathbf{C}_\infty$. We note each Q_r is $\mathfrak{C}_\infty \setminus \mathfrak{C}_r$ measurable. Let $\mathfrak{C}^*_\infty$ denote the smallest σ-field in $\mathbf{C}_\infty$ *wrt* which $Q_r,\ r = 1,\ 2, \ldots$ are measurable. □

Theorem 3.1. *(i) Fix* $f \in C_\infty$*. Then* $A = B$ *where*

$$\begin{aligned} A = \cup_{r=1}^{\infty}\{g\ :\ g \in \boldsymbol{C}_\infty.\ d_r(f,\ g) > \lambda\} \text{ and} \\ B = \{g\ :\ g \in \boldsymbol{C}_\infty,\ d(f,\ g) > \lambda\} \end{aligned} \quad (7)$$

(ii) $\mathfrak{C}^*_\infty = \mathfrak{C}_\infty$.

Proof.
(i) Let $g \in B$. If it is not admitted that $g \in A$, then $d_r(f,\ g) \le \lambda$ for each $r \ge 1$. Since $d_r(f,\ g)$ increases to $d(f,\ g)$, it follows that $d(f,\ g) \le \lambda$, a contradiction to the assumption $d(f,\ g) > \lambda$.
If now $g \in A$, then for some $r \ge 1$ (and hence for all large r) $d_r(f,\ g) > \lambda$. Since $d(f,\ g) \ge d_r(f,\ g) > \lambda$, it follows that $g \in A$.
(ii) That the Q_rs are continuous maps is easy to verify. Hence we conclude $\mathfrak{C}^*_\infty \subset \mathfrak{C}_\infty$. The reverse inclusion will stand proved if we show that every closed d-sphere $S(f;\ \lambda) = \{g\ :\ g \in \mathbf{C}_\infty,\ d(f,\ g) \le \lambda\}$ belongs to $\mathfrak{C}^*_\infty$. Now, since $d_r(f,\ g)$ increases to $d(f,\ g)$, $S(f;\ \lambda) = \{g\ :\ d_r(f,\ g) \le \lambda$ for every $r \ge 1\} = \cap_{r=1}^{\infty}\{g\ :\ d_r(f,\ g) \le \lambda\}$. Since $\{Q_r g\ :\ d_r(f,\ g) \le \lambda\} \in \mathbf{C}_r$, $\{g\ :\ d_r(f,\ g) \le \lambda\} \in \mathfrak{C}^*_\infty$. Hence $S(f;\ \lambda)$, being the intersection of a countable number of such sets, belongs to $\mathfrak{C}^*_\infty$. □

Theorem 3.2. *(i) If* $K \subset \boldsymbol{C}_\infty$ *is compact, then*

$$K = \cap_{r=1}^{\infty} Q_r^{-1} Q_r K. \quad (8)$$

(ii) For any set $A \subset \boldsymbol{C}_\infty$,

$$Q_{r+1}^{-1} Q_{r+1} A \subset Q_r^{-1} Q_r A. \quad (9)$$

Proof.
(i) That K is a subset of the rightside is trivial to see. To prove the converse, set $E = \cap_{r=1}^{\infty} Q_r^{-1} Q_r K$. Then the following relations hold.

$E \subset Q_r^{-1}Q_rK$ for every $r \geq 1 \Rightarrow Q_rE \subset Q_rK$ for every $r \geq 1 \Rightarrow Q_r^{-1}Q_rE \subset K$ for every $r \geq 1$.
Since E is compact we have, as observed earlier, $E \subset \cap_r Q_r^{-1}Q_rE$ and hence the required result follows.
(ii) Let $f \in Q_{r+1}^{-1}Q_{r+1}A$. Hence $Q_{r+1}f \in Q_{r+1}A$. There exists then $g \in A$ such that $Q_{r+1}f = Q_{r+1}g$. This implies $Q_rf = Q_rg$ and so $f \in Q_r^{-1}Q_rA$. □

Theorem 3.3. $\mu_r(Q_rK)$, $r = 1, 2, \ldots$ *is a monotonically decreasing sequence of real numbers.*

Proof.
Let T_r map $\mathbf{C}_{r+1}$ on to $\mathbf{C}_r$ according to the following scheme. $T_rQ_{r+1}g = Q_rg$. Thus $T_rQ_{r+1} = Q_r$. Recall Wiener measure μ_r is defined on $\mathfrak{C}_r$, $r = 1, 2, \ldots$. Both $\mu_{r+1}T_r^{-1}$ and μ_r are measures defined on $\mathfrak{C}_r$. Their finite dimensional distributions are the same. Hence the two are identical (ref. Remark 4). We then have (using the formula for change of variables in an integral (ref. Theorem 2.3.6, p91,[2]), $\mu_r(Q_rK) = \int\limits_{Q_rK} d\mu_r = \int\limits_{Q_rK} d\mu_{r+1}T_r^{-1} = \int\limits_{T_r^{-1}Q_rK} d\mu_{r+1} \geq \int\limits_{Q_{r+1}K} d\mu_{r+1}$.
We see from this that $\mu_r(Q_rK)$ is a monotonically decreasing sequence of real numbers.

For $K \in \mathfrak{C}_\infty$, K compact, define

$$\mu_\infty K = \lim_{r \text{ increases to } \infty} \mu_r(Q_rK) \tag{10}$$

and for arbitrary $A \in \mathfrak{C}_\infty$, define

$$\mu_\infty A = \sup_{K \subset A,\ K \text{ compact}} \mu_\infty K \tag{11}$$

and proceed as in the construction of the measure μ_a, use (10) and arrive at a countably additive finite measure μ_∞ which is finite and ≤ 1, by(10) and (11).
That μ_∞ is a probability measure will follow if we show that $\mu_\infty \mathbf{C}_\infty = 1$.
Consider the Hölder space $H_{\alpha,\infty}$ and define, for $x, y \in H_{\alpha,\infty}$ the metric

$$d_{\alpha,\infty}(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0\leq t,s\leq n; 0<|t-s|<1} \frac{|x(t)-y(t)-x(s)+y(s)|}{|t-s|^\alpha}}{1 + \sup_{0\leq t,s\leq n; 0<|t-s|<1} \frac{|x(t)-y(t)-x(s)+y(s)|}{|t-s|^\alpha}}$$

Also define

$$d^*_{\alpha,\infty}(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0\leq t,s\leq n; 0<|t-s|<1} \frac{|x(t)-y(t)|}{|t-s|^\alpha}}{1 + \sup_{0\leq t,s\leq n; 0<|t-s|<1} \frac{|x(t)-y(t)|}{|t-s|^\alpha}}.$$

Note that $d^*_{\alpha,\infty}(x,y) \leq d_{\alpha,\infty}(x,y) \leq 2d^*_{\alpha,\infty}(x,y)$. Further define on $H_{\alpha,\infty}$ another metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0\leq t\leq n} |x(t)-y(t)|}{1 + \sup_{0\leq t\leq n} |x(t)-y(t)|}.$$

Note that $d(x,y) \leq d^*_{\alpha,\infty}(x,y) \leq d_{\alpha,\infty}(x,y)$ Denote by ϑ the null element.

i.e., the function that is identically zero. Since $S^*(\lambda) = \{f : d(f, \vartheta) \leq \lambda\}$ increases to $\mathbf{C}_\infty$ as λ increases to ∞, it is sufficient to show that, given $\varepsilon > 0$, a λ can be found such that $\mu_\infty\big(S^*(\lambda)\big) > 1 - \varepsilon$.
Define $\mathbb{S}^*_\alpha(\lambda) = \{x : x \in H_{\alpha,\infty}, d_{\alpha,\infty}(\vartheta, x) \leq \lambda\}$. Since $d(x, y) \leq d_{\alpha,\infty}(x, y)$, $\mathbb{S}^*_\alpha(\lambda) \subset S^*(\lambda)$. Hence it is enough to find a λ such that $\mu_\infty(\mathbb{S}^*_\alpha(\lambda)) > 1 - \varepsilon$. Since $\mathbb{S}^*_\alpha(\lambda)$ is a compact set, $\mu_\infty(\mathbb{S}^*_\alpha(\lambda)) = \lim_{r\to\infty} \mu_r(Q_r\mathbb{S}^*_\alpha(\lambda))$ by(10). Take $r = [\lambda]$. The arguments in the proof of Theorem 2.3 apply and we get $\mu_\infty(\mathbb{S}^*_\alpha(\lambda)) \geq e^{-r/\lambda^2} \geq e^{-1/\lambda} \to 1$ as $\lambda \to \infty$. □

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Authors' contribution: The first author solely developed the alternate proof in Sub-section 2.1 and Section 3.

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4. Appendix: Compactness of $\mathbb{S}_{\alpha,a}(\lambda)$.

Recall

$$\mathbb{S}_{\alpha,a}(\lambda) = \{x : \|x\|_\alpha = \sup_{0<s,t\leq a;\ 0<|s-t|<1} \frac{|x(t) - x(s)|}{|s-t|^\alpha} \leq \lambda\}.$$

We shall show that $\mathbb{S}_{\alpha,a}(\lambda)$ is a compact subset of $\mathbf{C_a}$. To this end we shall show (a) that it is bounded, (b) that it is closed and (c) that it is uniformly equicontinuous.
Let $\alpha < \beta$. Then $\mathbb{S}_{\beta,a}(\lambda) \subset \mathbb{S}_{\alpha,a}(\lambda)$.

(b) Consider $u \in H_\alpha^0,\ u_n \in \mathbb{S}_{\beta,a}(\lambda)$ and let $\|u_n - u\|_\alpha \to 0$ as $n \to \infty$.

$$\sup_{0<s,t\le a;\ 0<|s-t|<1} \frac{|u(t)-u(s)|}{|s-t|^\beta} \le$$

$$\le \left(\sup_{0<s,t\le a;\ 0<|s-t|<1} \frac{|u(t)-u(s)|}{|s-t|^\alpha} \right)^{\beta/\alpha} \left(\sup_{0<s,t\le a;\ 0<|s-t|<1} |u(t)-u(s)| \right)^{1-\frac{\beta}{\alpha}}$$

$$= \lim_{n\to\infty} \left(\sup_{0<s,t\le a;\ 0<|s-t|<1} \frac{|u_n(t)-u_n(s)|}{|s-t|^\alpha} \right)^{\beta/\alpha} \left(\sup_{0<s,t\le a;\ 0<|s-t|<1} |u(t)-u(s)| \right)^{1-\frac{\beta}{\alpha}}$$

$$\le \limsup_{n\to\infty} \left(\sup_{0<s,t\le a;\ 0<|s-t|<1} \frac{|u_n(t)-u_n(s)|}{|s-t|^\beta} \right) \times$$

$$\limsup_{n\to\infty} \sup_{0<s,t\le a;\ 0<|s-t|<1} \left(|u_n(t)-u_n(s)|^{\frac{\beta}{\alpha}-1} |u(t)-u(s)|^{1-\frac{\beta}{\alpha}} \right)$$

$$\le \limsup_{n\to\infty} \|u_n\|_\beta \le \lambda.$$

This shows that $u \in \mathbb{S}_{\beta,a}$. i.e., $\mathbb{S}_{\beta,a}(\lambda)$ is a closed subset of H_α^0.
(c) Finally we prove that $\mathbb{S}_{\alpha,a}$ is uniformly equicontinuous in H_α^0. Let $x \in \mathbb{S}_{\alpha,a}(\lambda)$ and let $\delta > 0$ be arbitrary. Consider , for $\alpha < \beta$

$$\sup_{0\le s,t\le a;\ |t-s|\le\delta} \frac{|x(t)-x(s)|}{|t-s|^\alpha} = \sup_{0\le s,t\le a;\ |t-s|\le\delta} \frac{|x(t)-x(s)|}{|t-s|^\beta} \times |t-s|^{\beta-\alpha} \le \lambda\, \delta^{\beta-\alpha}.$$

Since δ is arbitrary and since the above step holds for all $x \in \mathbb{S}_{\alpha,a}$ uniform equicontinuity follows.
We thus have compactness of $\mathbb{S}_{\alpha,a}$.